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Determinant Type Differential Operators on Homogeneous Siegel Domains

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INTRODUCTION

In harmonic analysis on classical domains of matrices, the differential operator whose symbol is the determinant polynomial plays important roles. Particularly, the operator is substantial in study of invariant Hilbert spaces of holomorphic functions on the domain [1, 2, 7, 14, 15, 21]. Considering the Siegel domain realization of a certain symmetric domain with Fourier-analytic methods, Jakobsen and Vergne [15] show that a unitary representation of a semisimple Lie group is realized on a Hilbert space in the kernel of the wave operator, which corresponds to the determinant of a Hermitian 2×2 matrix, and that this differential operator intertwines some unitary representations of the group. In [14] this kind of equivariance property is investigated for powers of the differential operators corresponding to the determinants of $n\times n$ symmetric and Hermitian matrices. Arazy and Upmeier [2] (see also [1]) attain more general results from another approach, the Peter–Weyl theory for the maximal compact subgroups with the bounded realizations of symmetric domains.

In this paper, following the Fourier-analytic approach, we obtain analogues of these results in the framework of analysis on a homogeneous (not necessarily symmetric) Siegel domain $\mathscr{D}$ on which a split solvable Lie

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group G acts simply transitively by affine transformations. We are concerned with unitary representations of G realized on Hilbert spaces of holomorphic functions on $\mathscr{D}$. Such Hilbert spaces are closely related to orbits in the closure of the dual cone through the Fourier–Laplace transform (see [4, 12, 17, 19]). In the present work, we describe these orbits algebraically by making use of "determinant type polynomials" associated to the dual cone, which are generalizations of minors of real symmetric matrices introduced in [13]. Then the corresponding differential operators turn out to be the annihilators of the representation spaces or the intertwining operators between the unitary representations of G.

Although our work deals with the general homogeneous Siegel domains, we assume in this introduction that $\mathscr{D}$ is of tube type so that the essential ideas are presented simply. Let Ω be a homogeneous cone in a real vector space V and $H \subset GL(V)$ a split solvable linear Lie group acting on Ω simply transitively. The group H acts also simply transitively on the dual cone Ω^* in V^* by the contragredient action. Let $\mathscr{D}$ be the complex domain $V + i\Omega$ in the complexification of V. Then the semidirect product $G := V \rtimes H$ acts on $\mathscr{D}$ simply transitively by $(v, t) \cdot (x + iy) := v + t \cdot x + it \cdot y$ ($x, v \in V$, $y \in \Omega$, $t \in H$). Denote by $\mathscr{H}(\mathscr{D})$ the space of holomorphic functions on $\mathscr{D}$, and for a one-dimensional representation $\chi \colon G \to \mathbb{C}$, let π_χ be the representation of G on $\mathscr{H}(\mathscr{D})$ given by

$$\pi_\chi(g)\, F(z) := \chi(g)\, F(g^{-1} \cdot z) \qquad (g \in G, z \in \mathscr{D}, F \in \mathscr{H}(\mathscr{D})).$$

We consider a subspace $\mathscr{H}_\chi \subset \mathscr{H}(\mathscr{D})$ satisfying two conditions:

(i) $\mathscr{H}_\chi$ has a Hilbert space structure with reproducing kernel,

(ii) $(\pi_\chi, \mathscr{H}_\chi)$ is a unitary representation of G.

It is shown in [12] that, if $\mathscr{H}_\chi \neq \{0\}$, there exist an H-orbit $\mathcal{O}_\chi^*$ in the closure $\overline{\Omega^*}$ and a positive measure $d\nu_\chi$ on $\mathcal{O}_\chi^*$ such that one has a unitary isomorphism $\Phi_\chi \colon L^2(\mathcal{O}_\chi^*, d\nu_\chi) \to \mathscr{H}_\chi$, where

$$\Phi_\chi f(z) := \int_{\mathcal{O}_\chi^*} e^{i\langle z, \xi\rangle} f(\xi)\, d\nu_\chi(\xi) \qquad (f \in L^2(\mathcal{O}_\chi^*, d\nu_\chi), z \in \mathscr{D}). \tag{0.1}$$

Concerning the orbit $\mathcal{O}_\chi^*$, we show the following statement.

THEOREM A. *One can take irreducible polynomials* $\phi_1, \phi_2, ..., \phi_d$ *and* $\psi_1, \psi_2, ..., \psi_m$ *on* V^* *with the following three properties*:

(i) *The* H*-orbit* $\mathcal{O}_\chi^*$ *is described as*

$$\mathcal{O}_\chi^* = \{\xi \in V^*; \phi_\alpha(\xi) > 0 \ (\alpha = 1, ..., d), \psi_k(\xi) = 0 \ (k = 1, ..., m)\},$$

(ii) *For* $\alpha = 1, ..., d$, *the restriction of* ϕ_α *to* $\mathcal{O}_\chi^*$ *is H-relatively invariant,*

(iii) *If an H-relatively invariant function* ϕ *on* $\mathcal{O}_\chi^*$ *can be extended to a polynomial function on* V^*, *then* ϕ *is expressed as a product of powers of* $\phi_\alpha|_{\mathcal{O}_\chi^*}$ $(\alpha = 1, ..., d)$ *up to multiplication by a constant.*

The polynomials ϕ_α and ψ_k are chosen from the determinant type polynomials associated to the cone Ω^*.

For a polynomial φ on V^*, let $\varphi(\partial_z)$ be the differential operator such that $\varphi(\partial_z)\, e^{\langle z, \xi \rangle} = \varphi(\xi)\, e^{\langle z, \xi \rangle}$ $(\xi \in V^*)$. Then Theorem A is transformed by (0.1) to the statement concerning analysis on the Siegel domain $\mathcal{D}$.

THEOREM B. (i) *A function* $F \in \mathcal{H}_\chi$ *satisfies the differential equations* $\psi_k(-i\partial_z)\, F(z) = 0$ $(k = 1, ..., m)$.

(ii) *For* $\alpha = 1, ..., d$, *the differential operator* $\phi_\alpha(-i\partial_z)$ *induces an intertwining map from* $(\pi_\chi, \mathcal{H}_\chi)$ *to some other representation* $(\pi_{\chi'}, \mathcal{H}_{\chi'})$.

(iii) *If a differential operator Z induces an intertwining map from* $(\pi_\chi, \mathcal{H}_\chi)$ *to some* $(\pi_{\chi'}, \mathcal{H}_{\chi'})$, *then the same map is given by a product of powers of* $\phi_\alpha(-i\partial_z)$ $(\alpha = 1, ..., d)$ *up to constant multiple.*

Let us explain the organization of this paper. In the first part, Sections 1 to 3, we investigate the orbit structure of the closure of the general homogeneous cone Ω instead of its dual cone Ω^*. The basic tool is an algebra structure, *clan*, defined on the ambient vector space V of the cone Ω. In Section 1, as representatives of the H-orbits in the closure $\bar{\Omega}$, we take idempotents of the clan which are labeled by subsets I of $\{1, 2, ..., r\}$ (r is the rank of Ω) as E_I in (1.6). Then we associate each of the idempotents E_I to a lower dimensional homogeneous cone (*subcone*) Ω^I, and show in Proposition 1.2 that the H-orbit $\mathcal{O}_I$ through E_I has a natural fiber bundle structure with Ω^I its base space.

In Section 2, introducing the notion of determinant type polynomials, we find the generators of the set of H-relatively invariant polynomials on the orbit $\mathcal{O}_I$ in the sense of Theorem A (ii) and (iii). Since the fibration of $\mathcal{O}_I$ reduces the problem to the determination of the basic relative invariants associated to the subcone Ω^I (see Lemma 2.1), we give the solution in Theorem 2.3, utilizing results in [13]. The orbit $\mathcal{O}_I$ is described in Section 3 by using the determinant type polynomials as in Theorem A(i).

In the second part, Section 4, we are engaged in study of analysis on homogeneous Siegel domains. Since there exists the one-to-one correspondence between normal *j*-algebras and homogeneous Siegel domains established by Piatetskii–Shapiro [16], we start the argument with a normal *j*-algebra $\mathfrak{g}$, and consider the Siegel domain $\mathcal{D}$ constructed from $\mathfrak{g}$. Subsection 4.1 is devoted to presentation of this construction and some basic definitions. In Subsection 4.2, after a review of results in [12] about the unitary representations π_χ of G, we investigate intertwining operators between them. We show in Proposition 4.5 that, if two representations $(\pi_\chi, \mathscr{H}_\chi)$ and $(\pi_{\chi'}, \mathscr{H}_{\chi'})$ are equivalent, then the unitary intertwining operator $U: \mathscr{H}_\chi \to \mathscr{H}_{\chi'}$ is transferred by means of Φ_χ and $\Phi_{\chi'}$ to a multiplication operator by a relatively invariant function on the orbit $\mathcal{O}_\chi^*$. In Subsection 4.3, our results obtained in Sections 2 and 3 are applied to the investigation of the representations $(\pi_\chi, \mathscr{H}_\chi)$ of G. We describe the determinant type polynomials associated to the dual cone in terms of the normal j-algebra $\mathfrak{g}$, so that Theorem A is shown by applying Theorems 2.3 and 3.2 to that situation (see (4.14)). Then, we derive Theorem B (Theorem 4.6) from this and Proposition 4.5.

In concluding the introduction, we shall comment on the case that the Siegel domain $\mathscr{D}$ is symmetric. In this case, Ω is necessarily a symmetric cone [18, Theorem V.3.5]. Then the determinant type polynomials associated to Ω coincide with the determinants of the Jordan algebras associated to the subcones (see Remarks 2.4 and 4.7).

1. ORBIT STRUCTURE OF THE CLOSURE OF A HOMOGENEOUS CONE

Let V be a real vector space and Ω an open convex cone in V containing no line. We assume that the cone Ω is homogeneous, that is, the linear automorphism group $\mathrm{GL}(\Omega)$ of Ω acts on Ω transitively. Following Vinberg [20], we take a split solvable Lie subgroup $H \subset \mathrm{GL}(\Omega)$ which acts on Ω simply transitively, and introduce an algebra structure on the vector space V as follows. Let $\mathfrak{h} \subset \mathrm{End}(V)$ be the Lie algebra of H and fix a point E in Ω. Then we have the linear isomorphism $\mathfrak{h} \ni L \mapsto L \cdot E \in V$ obtained by differentiating the orbit map $H \ni t \mapsto t \cdot E \in \Omega$. Thus, for an element $x \in V$, there exists a unique $L_x \in \mathfrak{h}$ for which $L_x \cdot E = x$. We define a bilinear multiplication $\triangle$ on V by $x \triangle y := L_x \cdot y \in V$ $(x, y \in V)$. The algebra $(V, \triangle)$ is called *the clan* of the homogeneous cone Ω [20, Chap. 2]. This algebra has the following property:

$$[L_x, L_y] = L_{x \triangle y - y \triangle x} \qquad (x, y \in V). \tag{1.1}$$

By [20, Chap. 2, Proposition 8] we have *a normal decomposition* of the clan V,

$$V = \sum_{1 \leqslant k \leqslant m \leqslant r}^{\oplus} V_{mk}, \tag{1.2}$$

where V_{kk} $(k=1, ..., r)$ is the one-dimensional subspace spanned by an idempotent E_k, and V_{mk} consists of the elements $x \in V$ such that

$$c \vartriangle x = (1/2)(c_m + c_k)\, x \qquad \text{and} \qquad x \vartriangle c = c_k x \tag{1.3}$$

for all $c = \sum_{i=1}^{r} c_i E_i \in V$ $(c_1, ..., c_r \in \mathbb{R})$. The number r is called the *rank* of the cone Ω. Let E^* be the linear form on V given by

$$\left\langle \sum_{k=1}^{r} x_{kk} E_k + \sum_{1 \leqslant k < m \leqslant r} X_{mk},\, E^* \right\rangle := \sum_{k=1}^{r} x_{kk} \qquad (x_{kk} \in \mathbb{R},\, X_{mk} \in V_{mk}). \tag{1.4}$$

We see from [20, p. 376] that the bilinear form

$$(x \mid y) := \langle x \vartriangle y, E^* \rangle / 2 \qquad (x, y \in V) \tag{1.5}$$

defines an inner product on V and that the normal decomposition (1.2) of V is orthogonal with respect to this $(\cdot \mid \cdot)$.

For integers $a \leqslant b$, we denote by $\lfloor a, b \rfloor$ the set $\{a, a+1, ..., b\}$. Let I be a subset of $\lfloor 1, r \rfloor$. Put

$$E_I := \sum_{i \in I} E_i \in V, \tag{1.6}$$

and let $\mathcal{O}_I$ be the H-orbit $H \cdot E_I$ in V.

PROPOSITION 1.1 [11, Theorem 3.5]. *One has the following H-orbit decomposition*:

$$\bar{\Omega} = \bigsqcup_{I \subset \lfloor 1, r \rfloor} \mathcal{O}_I.$$

Let us investigate actions of subgroups of H on the orbits $\mathcal{O}_I$. We put $A_k := L_{E_k} \in \mathfrak{h}$ $(k=1, ..., r)$ and $\mathfrak{h}_{mk} := \{L_x;\, x \in V_{mk}\} \subset \mathfrak{h}$ $(1 \leqslant k < m \leqslant r)$, so that we have the decomposition $\mathfrak{h} = \sum_{1 \leqslant k \leqslant r}^{\oplus} \mathbb{R} A_k \oplus \sum_{1 \leqslant k < m \leqslant r}^{\oplus} \mathfrak{h}_{mk}$. We set

$$\mathfrak{h}(\mathcal{O}_I) := \sum_{k \in I}^{\oplus} \left(\mathbb{R} A_k \oplus \sum_{m > k}^{\oplus} \mathfrak{h}_{mk} \right).$$

Then $\mathfrak{h}(\mathcal{O}_I)$ is a subalgebra of $\mathfrak{h}$, and the corresponding subgroup $H(\mathcal{O}_I) := \exp \mathfrak{h}(\mathcal{O}_I)$ of H acts on the orbit $\mathcal{O}_I$ simply transitively [11, Lemma 3.3].

Setting $A^I := L_{E_I}$, we define $\mathfrak{h}_\mu(\mathcal{O}_I)$ ($\mu \in \mathbb{R}$) to be the eigenspace $\{T \in \mathfrak{h}(\mathcal{O}_I); [A^I, T] = \mu T\}$ of $\mathrm{ad}(A^I)|_{\mathfrak{h}(\mathcal{O}_I)}$. From (1.1) and (1.3) we get the decomposition $\mathfrak{h}(\mathcal{O}_I) = \mathfrak{h}_{-1/2}(\mathcal{O}_I) \oplus \mathfrak{h}_0(\mathcal{O}_I)$ with

$$\mathfrak{h}_0(\mathcal{O}_I) = \sum_{k \in I}^{\oplus} \left(\mathbb{R}A_k \oplus \sum_{m > k,\, m \in I}^{\oplus} \mathfrak{h}_{mk} \right), \tag{1.7}$$

$$\mathfrak{h}_{-1/2}(\mathcal{O}_I) = \sum_{k \in I}^{\oplus} \sum_{m > k,\, m \notin I}^{\oplus} \mathfrak{h}_{mk}. \tag{1.8}$$

We write $\mathfrak{h}^I$ and $\mathfrak{n}^I$ for $\mathfrak{h}_0(\mathcal{O}_I)$ and $\mathfrak{h}_{-1/2}(\mathcal{O}_I)$, respectively. Clearly

$$[\mathfrak{h}_\mu(\mathcal{O}_I), \mathfrak{h}_{\mu'}(\mathcal{O}_I)] \subset \mathfrak{h}_{\mu+\mu'}(\mathcal{O}_I),$$

so that $\mathfrak{h}^I$ is a Lie subalgebra, and $\mathfrak{n}^I$ a commutative ideal of $\mathfrak{h}(\mathcal{O}_I)$. Thus, putting $H^I := \exp \mathfrak{h}^I$ and $N^I := \exp \mathfrak{n}^I$, we have $H(\mathcal{O}_I) = N^I \rtimes H^I$. For $\lambda \in \mathbb{R}$, let $V(\lambda; A^I)$ be the eigenspace $\{x \in V; A^I \cdot x = \lambda x\}$ of the operator $A^I \in \mathfrak{h} \subset \mathrm{End}(V)$. Then $V = V(1; A^I) \oplus V(1/2; A^I) \oplus V(0; A^I)$. In fact, we have by (1.3)

$$V(1; A^I) = \sum_{\substack{m > k \\ m \in I,\, k \in I}}^{\oplus} V_{mk}, \qquad V(0; A^I) = \sum_{\substack{m > k \\ m \notin I,\, k \notin I}}^{\oplus} V_{mk}, \tag{1.9}$$

and

$$V(1/2; A^I) = \sum_{k \in I}^{\oplus} \left(\sum_{m > k,\, m \notin I}^{\oplus} V_{mk} \oplus \sum_{i < k,\, i \notin I}^{\oplus} V_{ki} \right).$$

It is immediate from the definitions that

$$\mathfrak{h}_\mu(\mathcal{O}_I) \cdot V(\lambda; A^I) \subset V(\mu + \lambda; A^I). \tag{1.10}$$

We write V^I for the subspace $V(1, A^I)$ of V. Then $(V^I, \triangle)$ is a subalgebra of $(V, \triangle)$ with E_I a unit element, and the H^I-orbit $\Omega^I := H^I \cdot E_I \subset V^I$ is the homogeneous cone whose clan is $(V^I, \triangle)$; see [13, Sect. 4]. We call this Ω^I the *subcone* corresponding to the idempotent E_I. Let $P_I \colon V \to V^I$ be the orthogonal projection with respect to the inner product (1.5), and denote by x_I the image $P_I(x)$ of $x \in V$.

PROPOSITION 1.2. (i) *One has*

$$P_I(nt \cdot x) = t \cdot P_I(x) \qquad (x \in V,\, n \in N^I,\, t \in H^I).$$

(ii) *The projection* P_I *maps the* H*-orbit* $\mathcal{O}_I$ *onto the subcone* Ω^I.

(iii) *For* $x \in \Omega^I$, *the set* $\mathcal{O}_I \cap P_I^{-1}(x)$ *equals the* N^I*-orbit through* x.

Proof. By (1.10), the group $H^I = \exp \mathfrak{h}_0(\mathcal{O}_I)$ preserves each of the subspaces V^I, $V(1/2; A^I)$ and $V(0; A^I)$. Thus the projection P_I is H^I-equivariant, that is,

$$P_I(t \cdot x) = t \cdot P_I(x) \qquad (x \in V, t \in H^I). \tag{1.11}$$

On the other hand, we see also from (1.10) that

$$\begin{gathered} \mathfrak{n}^I \cdot V^I \subset V(1/2; A^I), \qquad \mathfrak{n}^I \cdot V(1/2; A^I) \subset V(0; A^I), \\ \mathfrak{n}^I \cdot V(0; A^I) = \{0\}, \end{gathered} \tag{1.12}$$

which lead us to

$$P_I(n \cdot x) = P_I(x) \qquad (x \in V, n \in N^I).$$

Thus the assertion (i) is verified. The assertions (ii) and (iii) are clear from (i) because the group $N^I \rtimes H^I$ (resp. H^I) acts on $\mathcal{O}_I$ (resp. Ω^I) simply transitively. ∎

2. RELATIVELY INVARIANT POLYNOMIALS ON THE ORBITS $\mathcal{O}_I$

As is shown in [11, Sect. 4], H-relatively invariant functions on the orbit $\mathcal{O}_I$ are parameterized by d-tuples ($d = \sharp I$) of complex numbers (see also (4.9)). Among them, we are concerned with the ones which can be extended to polynomial functions on V. In this section, generators of the set of such functions are determined.

A polynomial F on V is said to be H-relatively invariant on the orbit $\mathcal{O}_I$ if the restriction $F|_{\mathcal{O}_I}$ is an H-relatively invariant function on $\mathcal{O}_I$. In other words, there exists a one-dimensional representation $\chi: H \to \mathbb{C}$ such that the equality $F(t \cdot x) = \chi(t)\, F(x)$ holds for $x \in \mathcal{O}_I$ and $t \in H$, where χ is called *the multiplier* corresponding to F.

LEMMA 2.1. (i) *If a polynomial* F *is* H*-relatively invariant on* $\mathcal{O}_I$, *then one has* $F(x) = F(x_I)$ *for* $x \in \mathcal{O}_I$.

(ii) *For an* H^I*-relatively invariant polynomial* f *on* V^I, *the polynomial* F *on* V *given by* $F(x) := f(x_I)$ $(x \in V)$ *is* H*-relatively invariant on* $\mathcal{O}_I$.

Proof. (i) Since $\mathfrak{n}^I = [A^I, \mathfrak{n}^I] \subset [\mathfrak{h}, \mathfrak{h}]$ by definition, the restriction of any multiplier to the group N^I equals 1. Thus the assertion (i) follows from Proposition 1.2(iii).

(ii) Let $\tilde{\chi}: H^I \to \mathbb{C}$ be the multiplier corresponding to f. We see from [11, Sect. 4] that this $\tilde{\chi}$ is extended uniquely to a one-dimensional representation $\chi: H \to \mathbb{C}$ in such a way that the restriction of χ to the stabilizer H_{E_I} at E_I is equal to 1. On the other hand, for any $t \in H$ there exist unique $n \in N^I$, $t_1 \in H^I$, and $t_2 \in H_{E_I}$ such that $t = nt_1 t_2$ because $N^I \rtimes H^I$ acts simply transitively on $\mathcal{O}_I \simeq H/H_{E_I}$. Thus we have by Proposition 1.2(iii)

$$F(t \cdot E_I) = f(t_1 \cdot E_I) = \tilde{\chi}(t_1) f(E_I) = \chi(t) F(E_I),$$

whence the assertion follows. ∎

Lemma 2.1 reduces studies of relatively invariant polynomials on the orbits to investigations of ones on the ambient vector spaces of homogeneous cones. Now we recall results in [13], which determine the basic relative invariants under the action of H on the ambient space V of the cone Ω.

Let $\|\cdot\|$ be the norm on V given by $\|x\|^2 := (x \mid x)$ $(x \in V)$, see (1.5). For $x \in V$ and $i = 1, ..., r$, define $x^{(i)} = \sum_{k=1}^{r} x_{kk}^{(i)} E_k + \sum_{m>k} X_{mk}^{(i)} \in V$ by

$$\begin{aligned} x^{(1)} &:= x, \\ x_{kk}^{(i+1)} &:= x_{ii}^{(i)} x_{kk}^{(i)} - \|X_{ki}^{(i)}\|^2 \qquad (i < k \leqslant r), \\ X_{mk}^{(i+1)} &:= x_{ii}^{(i)} X_{mk}^{(i)} - X_{mi}^{(i)} \triangle X_{ki}^{(i)} \qquad (i < k < m \leqslant r), \end{aligned} \tag{2.1}$$

and put

$$D_k(x) := x_{kk}^{(k)} \in \mathbb{R} \qquad (k = 1, ..., r).$$

It is shown in [20] (see also [9]) that the polynomials D_k are H-relatively invariant. We define polynomials $\Delta_1, ..., \Delta_r$ inductively as follows: (a) $\Delta_1 := D_1$, (b) When $\Delta_1, ..., \Delta_{k-1}$ are determined, divide D_k by them as many times as possible, and let Δ_k be the remaining quotient. Namely, we have a factorization

$$D_k = \Delta_k \cdot (\Delta_1)^{a_{k,1}} \cdots (\Delta_{k-1})^{a_{k,k-1}} \tag{2.2}$$

with the conditions that

(i) $a_{k,1}, ..., a_{k,k-1}$ are non-negative integers,

(ii) Δ_k is not divisible by any of the polynomials $\Delta_1, ..., \Delta_{k-1}$.

PROPOSITION 2.2 [13, Theorem 2.2]. *The polynomials $\Delta_1, ..., \Delta_r$ are irreducible and H-relatively invariant. Moreover, every H-relatively invariant polynomial is expressed as a product of their powers up to constant multiple.*

We denote by D the polynomial D_r and call it *the composite determinant* associated to the cone Ω. Similarly, we define *the reduced determinant* Δ to be the polynomial Δ_r. Let D^I and Δ^I the composite and reduced determinant on V^I associated to the subcone Ω^I, and extend them to polynomials on V by $D^I(x) := D^I(x_I)$, $\Delta^I(x) := \Delta^I(x_I)$ $(x \in V)$. Then the basic relative invariants Δ_k on V are equal to $\Delta^{\lfloor 1, k \rfloor}$. More generally, if $I = \{i_1, i_2, ..., i_d\}$ with $1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant r$, then $\Delta^{I_1}, \Delta^{I_2}, ..., \Delta^{I_d}$ $(I_\alpha := \{i_1, i_2, ..., i_\alpha\})$ are the generators of the H^I-relatively invariant polynomials on V^I. The factorization (2.2) generalizes to

$$D^I = \Delta^I \cdot (\Delta^{I_1})^{c_1} \cdots (\Delta^{I_{d-1}})^{c_{d-1}} \tag{2.3}$$

with non-negative integers $c_1, ..., c_{d-1}$.

These observations together with Lemma 2.1 yield the following result.

THEOREM 2.3. *A polynomial F is H-relatively invariant on the orbit $\mathcal{O}_I$ if and only if there exist non-negative integers $a_1, a_2, ..., a_d$ $(d = \sharp I)$ and a constant $C \in \mathbb{C}$ for which the equality*

$$F = C \cdot (\Delta^{I_1})^{a_1} (\Delta^{I_2})^{a_2} \cdots (\Delta^{I_d})^{a_d}$$

holds on the orbit $\mathcal{O}_I$.

Remark 2.4. If the cone Ω is symmetric, the ambient vector space V has a Jordan algebra structure [7]. In this case, each subcone Ω^I is also symmetric and V^I is a Jordan subalgebra of V. The reduced determinant Δ^I coincides with the Jordan-determinant of V^I.

3. ALGEBRAIC DESCRIPTIONS OF THE ORBITS $\mathcal{O}_I$

In this section, we characterize the orbit $\mathcal{O}_I$ in terms of the determinant type polynomials D^J and Δ^J $(J \subset \lfloor 1, r \rfloor)$. For $k \in \lfloor 1, r \rfloor \backslash I$ and $m \in \lfloor k, r \rfloor$, we put $I^{km} := (I \cap \lfloor 1, k \rfloor) \cup \{k\} \cup \{m\}$. If there exists an α for which $i_\alpha < k < i_{\alpha+1}$, we have

$$I^{kk} = \{i_1, i_2, ..., i_\alpha, k\}, \qquad I^{km} = \{i_1, i_2, ..., i_\alpha, k, m\} \qquad (m > k). \tag{3.1}$$

Set $\mathcal{N}(I) := \{I^{km}; k \notin I, m \in \lfloor k, r \rfloor\}$ and recall that $d = \sharp I$.

PROPOSITION 3.1. *The H-orbit $\mathcal{O}_I$ is described as*

$$\mathcal{O}_I = \left\{ x \in V;\ \begin{matrix} D^{I_\alpha}(x) > 0 & (\alpha = 1, ..., d) \\ D^J(x) = 0 & (J \in \mathcal{N}(I)) \end{matrix} \right\}. \tag{3.2}$$

Proof. We prove the proposition by induction on the rank r. The case $r = 1$ is obvious. For $r \geqslant 2$, assume that the assertion holds for the rank $r - 1$ cone $\Omega^{\lfloor 2, r \rfloor} \subset V^{\lfloor 2, r \rfloor}$.

We first prove the statement for the case $1 \notin I$. Since $H(\mathcal{O}_I) = N^I \rtimes H^I$ is contained in $H^{\lfloor 2, r \rfloor}$ in this case, $\mathcal{O}_I$ equals the $H^{\lfloor 2, r \rfloor}$-orbit in $V^{\lfloor 2, r \rfloor}$ through $E_I \in V^{\lfloor 2, r \rfloor}$, so that the induction hypothesis says

$$\mathcal{O}_I = \left\{ x \in V^{\lfloor 2, r \rfloor};\ \begin{matrix} D^{I_\alpha}(x) > 0 & (\alpha = 1, ..., d) \\ D^J(x) = 0 & (J \in \mathcal{N}'(I)) \end{matrix} \right\}, \tag{3.3}$$

where

$$\mathcal{N}'(I) := \{ J \in \mathcal{N}(I);\ J \subset \lfloor 2, r \rfloor \}. \tag{3.4}$$

Here we remark that

$$\mathcal{N}(I) = \mathcal{N}'(I) \cup \{ I^{11}, I^{12}, ..., I^{1r} \} \tag{3.5}$$

and that $I^{11} = \{1\}, I^{1k} = \{1, k\}$ $(k > 1)$. On the other hand, since we have by (2.1)

$$D^{I^{1k}}(x) = \begin{cases} x_{11} & (k = 1), \\ x_{11} x_{kk} - \|X_{k1}\|^2 & (k = 2, ..., r), \end{cases}$$

the condition that $D^J(x) = 0$ for $J = I^{1k}$ $(k = 1, ..., r)$ is equivalent to that $x_{11} = 0$ and $X_{k1} = 0$ $(k = 2, ..., r)$, which mean $x \in V^{\lfloor 2, r \rfloor}$. Thus we obtain from (3.3)

$$\mathcal{O}_I = \left\{ x \in V;\ \begin{matrix} D^{I_\alpha}(x) > 0 & (\alpha = 1, ..., d) \\ D^J(x) = 0 & (J \in \mathcal{N}'(I) \cup \{ I^{11}, ..., I^{1r} \}) \end{matrix} \right\},$$

which together with (3.5) shows (3.2).

Next we prove the statement for the case $1 \in I$. Put $I' := I \backslash \{1\}$. Then we see

$$\mathcal{N}'(I') = \{ J \backslash \{1\};\ J \in \mathcal{N}(I) \} \tag{3.6}$$

from the definition (3.4), and

$$I'_\alpha = I_{\alpha+1} \backslash \{1\} \qquad (\alpha = 1, ..., d-1). \tag{3.7}$$

Let $\mathcal{O}$ be the right-hand side of (3.2) and x an element of $\mathcal{O}$. Noting that $D^{I_1}(x) = x_{11} > 0$, we put

$$x' := \exp\left(-\sum_{m=2}^{r} L_{X_{m1}}/x_{11}\right) \cdot x \tag{3.8}$$

By [13, Lemma 4.2] we see that

$$x' = x_{11} E_1 + x'_{\lfloor 2, r \rfloor} \tag{3.9}$$

and that, for a set $J \subset \lfloor 1, r \rfloor$ containing 1,

$$D^{J \setminus \{1\}}(x'_{\lfloor 2, r \rfloor}) = (x_{11})^{-2\beta} D^J(x) \qquad (\beta := \sharp J - 2). \tag{3.10}$$

It follows from (3.6), (3.7), and (3.10) that

$$\begin{aligned} x_{11} > 0, \qquad & D^{I'_\alpha}(x'_{\lfloor 2, r \rfloor}) > 0 \quad (\alpha = 1, ..., d-1), \\ D^{J'}(x'_{\lfloor 2, r \rfloor}) = 0 \qquad & (J' \in \mathcal{N}'(I')). \end{aligned} \tag{3.11}$$

By induction hypothesis, the last two conditions mean $x'_{\lfloor 2, r \rfloor} \in \mathcal{O}_{I'} \subset V^{\lfloor 2, r \rfloor}$. Therefore $x' = x_{11} E_1 + x'_{\lfloor 2, r \rfloor} \in \mathcal{O}_I$, so that x also belongs to $\mathcal{O}_I$ by (3.8). Namely we have verified $\mathcal{O} \subset \mathcal{O}_I$. In order to prove the converse inclusion $\mathcal{O}_I \subset \mathcal{O}$, it suffices to show

Claim. If $x \in \mathcal{O}_I$, then one has $x_{11} > 0$ and $x'_{\lfloor 2, r \rfloor} \in \mathcal{O}_{I'}$, where x' be the element given by (3.8).

Indeed, for $x \in \mathcal{O}_I$ the claim implies (3.11) by induction hypothesis, so that $x \in \mathcal{O}$ by (3.10). We give here a sketch of the proof of the claim because a quite similar assertion is shown in the proof of [13, Lemma 4.2]. We see from [11, Proposition 2.5] that $x_{11} > 0$. Take $t' \in H$ for which $x' = t' \cdot E_I \in \mathcal{O}_I$. Then $x' = t' \cdot (E_1 + E_{I'}) = t' \cdot E_1 + t' \cdot E_{I'}$. Since $t' \cdot E_{I'}$ belongs to $V^{\lfloor 2, r \rfloor}$, the V_{m1}-components $(m > 1)$ of $t' \cdot E_1$ coincide with the ones of x', which equal 0 by (3.9). From this, we can deduce $t' \cdot E_1 = x_{11} E_1$. Thus we obtain $t' \cdot E_{I'} = x'_{\lfloor 2, r \rfloor}$, so that $x_{11} > 0$ and $x'_{\lfloor 2, r \rfloor} \in \mathcal{O}_{I'}$. ∎

Noting that $I^{km} = I^{kk} \cup \{m\}$ for $m > k$, and keeping (2.3) in mind, we define a subset $\mathcal{M}(I)$ of $\mathcal{N}(I)$ to be

$$\{I^{kk};\, k \notin I\} \cup \{I^{km};\, k \notin I,\, m > k,\, D^{I^{km}} \text{ is not divisible by } \Delta^{I^{kk}}\}.$$

We remark that $\mathcal{M}(I)$ depends on the homogeneous cone Ω, while $\mathcal{N}(I)$ depends only on the set I.

THEOREM 3.2. *The H-orbit $\mathcal{O}_I$ is described as*

$$\mathcal{O}_I = \left\{ x \in V; \begin{array}{ll} \Delta^{I_\alpha}(x) > 0 & (\alpha = 1, ..., d) \\ \Delta^J(x) = 0 & (J \in \mathcal{M}(I)) \end{array} \right\}. \tag{3.12}$$

Proof. Let $\mathcal{O}$ be the right-hand side of (3.12). It is easily seen from (2.3) that

$$\begin{aligned} & D^{I_1}(x) > 0, D^{I_2}(x) > 0, ..., D^{I_\alpha}(x) > 0 \\ \Leftrightarrow \quad & \Delta^{I_1}(x) > 0, \Delta^{I_2}(x) > 0, ..., \Delta^{I_\alpha}(x) > 0. \end{aligned}$$

When $i_\alpha < k < i_{\alpha+1}$, we have by (3.1) and (2.3)

$$\begin{aligned} D^{I^{kk}} &= \Delta^{I^{kk}} \cdot (\Delta^{I_1})^{c_1} (\Delta^{I_2})^{c_2} \cdots (\Delta^{I_\alpha})^{c_\alpha}, \\ D^{I^{km}} &= \Delta^{I^{km}} \cdot (\Delta^{I_1})^{d_1} (\Delta^{I_2})^{d_2} \cdots (\Delta^{I_\alpha})^{d_\alpha} \qquad (\text{if } I^{km} \in \mathcal{M}(I)), \end{aligned}$$

where c_β, d_β $(\beta = 1, ..., \alpha)$ are some non-negative integers. When $k < i_1$, we have

$$\begin{aligned} D^{I^{kk}} &= \Delta^{I^{kk}}, \\ D^{I^{km}} &= \Delta^{I^{km}} \qquad (\text{if } I^{km} \in \mathcal{M}(I)). \end{aligned}$$

These facts together with Proposition 3.2 tell us that if $x \in \mathcal{O}_I$, then $x \in \mathcal{O}$. In order to show the converse, it suffices to check that, if $\Delta^J(x) = 0$ for all $J \in \mathcal{M}(I)$, then $D^K(x) = 0$ for all $K \in \mathcal{N}(I)$. When $I^{km} \in \mathcal{M}(I)$, the equality $D^{I^{km}}(x) = 0$ follows from $\Delta^{I^{km}}(x) = 0$. On the other hand, when $I^{km} \in \mathcal{N}(I) \setminus \mathcal{M}(I)$, the polynomial $D^{I^{km}}$ is a multiple of $\Delta^{I^{kk}}$ by definition, so that one has $D^{I^{km}}(x) = 0$. Hence the theorem is verified. ∎

4. DIFFERENTIAL OPERATORS ON HOMOGENEOUS SIEGEL DOMAINS

In the second half of this paper, we apply the results about the determinant type polynomials to a study of analysis on homogeneous Siegel domains.

4.1. *Siegel domains and normal j-algebras*

Based on the one-to-one correspondence between normal *j*-algebras and homogeneous Siegel domains established by [16], we may assume without loss of generality that the Siegel domain in our consideration is defined through a normal *j*-algebra in the way explained below. First we recall the definition of normal j-algebras. Let $\mathfrak{g}$ be a real split solvable Lie algebra, j a linear operator on $\mathfrak{g}$ such that $j^2 = -\mathrm{id}_{\mathfrak{g}}$, and ω a linear form on $\mathfrak{g}$. The triple $(\mathfrak{g}, j, \omega)$ is called *a normal j-algebra* if the following two conditions are satisfied:

(i) $[Y_1, Y_2] + j[jY_1, Y_2] + j[Y_1, jY_2] - [jY_1, jY_2] = 0$ for all $Y_1, Y_2 \in \mathfrak{g}$,

(ii) the bilinear form $(Y_1 \mid Y_2)_\omega := \omega([Y_1, jY_2])$ defines a j-invariant inner product on $\mathfrak{g}$.

Let $\mathfrak{a}$ be the orthogonal complement of the subspace $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ with respect to the inner product $(\cdot \mid \cdot)_\omega$. Then $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$. Put $r := \dim \mathfrak{a}$, and for a linear form $\alpha \in \mathfrak{a}^*$, set

$$\mathfrak{g}_\alpha := \{ Y \in \mathfrak{g};\ [C, Y] = \alpha(C)\, Y \text{ for all } C \in \mathfrak{a} \}.$$

PROPOSITION 4.1 [16, Chap. 2, Sect. 3 and 5]. (i) *There is a linear basis* $\{A_1, ..., A_r\}$ *of* $\mathfrak{a}$ *such that if one puts* $E_l := -jA_l$, *then* $[A_k, E_l] = \delta_{kl} E_l$ $(k, l \in \lfloor 1, r \rfloor)$.

(ii) *Let* $\alpha_1, ..., \alpha_r$ *be the basis of* $\mathfrak{a}^*$ *dual to* $A_1, ..., A_r$. *Then one has a decomposition* $\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1/2) \oplus \mathfrak{g}(1)$ *with*

$$\mathfrak{g}(0) := \mathfrak{a} \oplus \sum_{1 \leqslant k < m \leqslant r}^{\oplus} \mathfrak{g}_{(\alpha_m - \alpha_k)/2}, \quad \mathfrak{g}(1/2) := \sum_{k=1}^{r\ \oplus} \mathfrak{g}_{\alpha_k/2},$$

$$\mathfrak{g}(1) := \sum_{k=1}^{r\ \oplus} \mathbb{R} E_k \oplus \sum_{1 \leqslant k < m \leqslant r}^{\oplus} \mathfrak{g}_{(\alpha_m + \alpha_k)/2}. \tag{4.1}$$

(iii) *One has* $[\mathfrak{g}(\mu), \mathfrak{g}(\nu)] \subset \mathfrak{g}(\mu + \nu)$ $(\mu, \nu = 0, 1/2, 1)$, *where* $\mathfrak{g}(\mu) := \{0\}$ *if* $\mu > 1$.

(iv) *If* $m > k$, *then* $j\mathfrak{g}_{(\alpha_m - \alpha_k)/2} = \mathfrak{g}_{(\alpha_m + \alpha_k)/2}$, *and the action of j is given by* $jY = -[Y, E_k]$ $(Y \in \mathfrak{g}_{(\alpha_m - \alpha_k)/2})$. *Moreover one has* $j\mathfrak{g}_{\alpha_k/2} = \mathfrak{g}_{\alpha_k/2}$.

The assertion (iii) tells us that the subspace $\mathfrak{g}(0)$ is a subalgebra of $\mathfrak{g}$ and that the corresponding Lie group $G(0) := \exp \mathfrak{g}(0)$ acts on $\mathfrak{g}(1/2)$ and $\mathfrak{g}(1)$ by the adjoint action. Put $E := E_1 + \cdots + E_r \in \mathfrak{g}(1)$. It is known [17, Theorem 4.15] that the $G(0)$-orbit $\Omega := G(0) \cdot E$ is a regular open convex cone in $\mathfrak{g}(1)$ on which the group $G(0)$ acts simply transitively.

Remark 4.2. The notation is compatible with the one in the previous sections. Indeed, since the clan of the cone $\Omega \subset \mathfrak{g}(1)$ is given as $x \triangle y := [jx, y]$ for $x, y \in \mathfrak{g}(1)$ (see [5, p. 536]), the elements E_k are idempotents of this clan by Proposition 4.1 and the corresponding normal decomposition coincides with the root space decomposition of $\mathfrak{g}(1)$ in (4.1).

The linear operator j preserves the subspace $\mathfrak{g}(1/2)$, so that j defines a complex structure on $\mathfrak{g}(1/2)$. Let $\mathfrak{g}(1)_{\mathbb{C}}$ be the complexification of $\mathfrak{g}(1)$, and define a $\mathfrak{g}(1)_{\mathbb{C}}$-valued Hermitian map Q on $\mathfrak{g}(1/2)$ by $Q(u, u') := ([ju, u'] + i[u, u'])/4$ $(u, u' \in \mathfrak{g}(1/2))$. Then Q is Ω-positive, that is, $Q(u, u) \in \overline{\Omega} \backslash \{0\}$ for all $u \in \mathfrak{g}(1/2) \backslash \{0\}$. Now we define the Siegel domain $\mathscr{D}$ corresponding to $(\mathfrak{g}, j, \omega)$ by $\mathscr{D} := \{(z, u) \in \mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1/2); \operatorname{Im} z - Q(u, u) \in \Omega\}$.

By Proposition 4.1(iii), the space $\mathfrak{n}(Q) := \mathfrak{g}(1) \oplus \mathfrak{g}(1/2)$ is a nilpotent subalgebra of $\mathfrak{g}$. Put $N(Q) := \exp \mathfrak{n}(Q)$. Then the solvable Lie group $G := \exp \mathfrak{g}$ is a semidirect product $N(Q) \rtimes G(0)$. We realize G as an affine transformation group on $\mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1/2)$ by setting

$$\begin{aligned} &\exp(x_0 + u_0)\, t_0 \cdot (z, u) \\ &\qquad := (t_0 \cdot z + x_0 + 2iQ(t_0 \cdot u, u_0) + iQ(u_0, u_0), t_0 \cdot u + u_0) \\ &\quad (x_0 \in \mathfrak{g}(1), u_0 \in \mathfrak{g}(1/2), t_0 \in G(0), (z, u) \in \mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1/2)). \end{aligned}$$

Then G acts on $\mathscr{D}$ simply transitively.

4.2. *Unitary Representations of G*

For $s = (s_1, ..., s_r) \in \mathbb{C}^r$, let χ_s be the one-dimensional representation of G such that

$$\chi_s\left(\exp\left(\sum_{k=1}^{r} c_k A_k\right)\right) = e^{s_1 c_1 + \cdots + s_r c_r} \qquad (c_1, ..., c_r \in \mathbb{R}). \tag{4.2}$$

Let $\mathscr{H}(\mathscr{D})$ be the space of holomorphic functions on $\mathscr{D}$. For $s \in \mathbb{C}^r$, we define a representation π_s on $\mathscr{H}(\mathscr{D})$ by

$$\pi_s(g) F(p) := \chi_{-s/2}(g) F(g^{-1} \cdot p) \qquad (g \in G, p \in D, F \in \mathscr{H}(\mathscr{D})). \tag{4.3}$$

We consider the subspace $\mathscr{H}_s(\mathscr{D})$ of $\mathscr{H}(\mathscr{D})$ satisfying the following two conditions:

(i) $\mathscr{H}_s(\mathscr{D})$ has a Hilbert space structure with reproducing kernel,

(ii) $(\pi_s, \mathscr{H}_s(\mathscr{D}))$ is a unitary representation of G.

We note that for each $s \in \mathbb{C}^r$ a non-zero $\mathscr{H}_s(\mathscr{D})$ is unique provided it exists, since the reproducing kernel is determined from the s ([12, Proposition 4.6]). For $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_r) \in \{0, 1\}^r$, put

$$q_k(\varepsilon) := \sum_{m > k} \varepsilon_m \dim \mathfrak{g}_{(\alpha_m - \alpha_k)/2} \qquad (k = 1, ..., r),$$

$$Z(\varepsilon) := \{\zeta = (\zeta_1, ..., \zeta_r) \in \mathbb{R}^r; \zeta_k = 0 \text{ for all } k \text{ such that } \varepsilon_k = 1\},$$

and for $\zeta \in Z(\varepsilon)$ let $\Theta(\varepsilon, \zeta)$ be the set

$$\left\{ s = (s_1, ..., s_r) \in \mathbb{C}^r;\ \begin{matrix} \Re s_k > q_k(\varepsilon)/2 & (\text{if } \varepsilon_k = 1) \\ s_k = q_k(\varepsilon)/2 - 2i\zeta_k & (\text{if } \varepsilon_k = 0) \end{matrix} \right\}. \tag{4.4}$$

PROPOSITION 4.3 [12, Theorems 4.8 and 5.3]. *A non-zero $\mathscr{H}_s(\mathscr{D})$ exists if and only if $s \in \bigsqcup_{\varepsilon \in \{0,1\}^r} \bigsqcup_{\zeta \in Z(\varepsilon)} \Theta(\varepsilon, \zeta)$. In this case, the representation $(\pi_s, \mathscr{H}_s(\mathscr{D}))$ is irreducible. Two representations $(\pi_s, \mathscr{H}_s(\mathscr{D}))$ and $(\pi_{s'}, \mathscr{H}_{s'}(\mathscr{D}))$ are equivalent if and only if s and s' belong to the same $\Theta(\varepsilon, \zeta)$.*

Let us describe the Hilbert space $\mathscr{H}_s(\mathscr{D})$ through the Fourier–Laplace transform. Let Ω^* be the dual cone of Ω, that is, $\Omega^* := \{\xi \in \mathfrak{g}(1)^*; \langle x, \xi \rangle > 0 \text{ for all } x \in \overline{\Omega} \backslash \{0\}\}$. The group $G(0)$ acts on Ω^* simply transitively by the contragredient action. For $\varepsilon = (\varepsilon_1, ..., \varepsilon_r) \in \{0, 1\}^r$, let E_ε^* be the linear form on $\mathfrak{g}(1)$ given by

$$\left\langle \sum_{k=1}^r x_{kk} E_k + \sum_{m>k} X_{mk}, E_\varepsilon^* \right\rangle := \sum_{k=1}^r \varepsilon_k x_{kk} \qquad (x_{kk} \in \mathbb{R},\ X_{mk} \in \mathfrak{g}_{(\alpha_m + \alpha_k)/2}). \tag{4.5}$$

Let $\mathcal{O}_\varepsilon^*$ be the $G(0)$-orbit in $\mathfrak{g}(1)^*$ through E_ε^*. Then the $G(0)$-orbit decomposition of the closure of the cone Ω^* is given as $\overline{\Omega^*} = \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathcal{O}_\varepsilon^*$. For $s \in \Theta(\varepsilon, \zeta)$, there exists a unique (up to constant multiple) measure $d\nu_s$ on $\mathcal{O}_\varepsilon^*$ such that

$$d\nu_s(t \cdot \xi) = \chi_{-\Re s}(t)\, d\nu_s(\xi) \qquad (t \in G(0),\ \xi \in \mathcal{O}_\varepsilon^*), \tag{4.6}$$

where $\Re s = (\Re s_1, ..., \Re s_r) \in \mathbb{R}^r$ [12, Theorem 2.3(iii)]. For $\xi \in \overline{\Omega^*}$, put $Q_\xi := 2\xi \circ Q$ and $N_\xi := \{v \in \mathfrak{g}(1/2); Q_\xi(v, v) = 0\}$. Then Q_ξ induces a positive definite Hermitian form on the quotient space $\mathfrak{g}(1/2)/N_\xi$. Let $\mathscr{F}_\xi$ be the Fock space on $\mathfrak{g}(1/2)/N_\xi$ with the reproducing kernel $e^{Q_\xi(\cdot, \cdot)}$ (see [3; 12, Sect. 3]). We regard $\mathscr{F}_\xi$ as a function space on $\mathfrak{g}(1/2)$ in what follows. For $s \in \Theta(\varepsilon, \zeta)$, we define $\mathscr{L}_s$ to be the space of measurable functions f on $\mathcal{O}_\varepsilon^* \times \mathfrak{g}(1/2)$ such that

(i) $f(\xi, \cdot) \in \mathscr{F}_\xi$ for almost all $\xi \in \mathcal{O}_\varepsilon^*$ with respect to the measure $d\nu_s$,

(ii) $\|f\|^2 := \int_{\mathcal{O}_\varepsilon^*} \|f(\xi, \cdot)\|_\xi^2\, d\nu_s(\xi) < \infty$.

Then $\mathscr{L}_s$ forms a Hilbert space, realizing the direct integral $\int_{\mathcal{O}_\varepsilon^*}^{\oplus} \mathscr{F}_\xi\, d\nu_s(\xi)$.

PROPOSITION 4.4 [12, Theorem 4.10]. *For $s \in \Theta(\varepsilon, \zeta)$, one has a unitary isomorphism Φ_s: $\mathscr{L}_s \ni f \mapsto F \in \mathscr{H}_s(\mathscr{D})$ given by*

$$F(z, u) := \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z, \xi \rangle} f(\xi, u)\, d\nu_s(\xi) \qquad ((z, u) \in D).$$

We define the so-called Fock representation τ_ξ of $N(Q)$ on $\mathscr{F}_\xi$ by

$$\tau_\xi(\exp(x_0+u_0))\,\phi(u) := e^{-i\langle x_0,\,\xi\rangle + Q_\xi(u,\,u_0) - Q_\xi(u_0,\,u_0)/2}\phi(u-u_0)$$

$$(\phi\in\mathscr{F}_\xi,\ u,\ u_0\in\mathfrak{g}(1/2),\ x_0\in\mathfrak{g}(1)),$$

and a unitary representation $\check{\pi}_s$ $(s\in\Theta(\varepsilon,\zeta))$ of G on $\mathscr{L}_s$ by

$$\check{\pi}_s(t)\,f(\xi,u) := \chi_{\bar{s}/2}(t)\,f(t^{-1}\cdot\xi,\,t^{-1}\cdot u) \qquad (t\in G(0)), \tag{4.7}$$

$$\check{\pi}_s(n)\,f(\xi,\cdot) := \tau_\xi(n)\,f(\xi,\cdot) \qquad (n\in N(Q)), \tag{4.8}$$

where $f\in\mathscr{L}_s$ and $\bar{s}:=(\bar{s}_1,\ldots,\bar{s}_r)\in\mathbb{C}^r$. Then the operator $\Phi_s\colon\mathscr{L}_s\to\mathscr{H}_s(\mathscr{D})$ is an intertwining operator between $\check{\pi}_s$ and π_s [12, Proposition 4.11].

Now we shall give an explicit description of the unitary intertwining operator between $(\check{\pi}_s,\mathscr{L}_s)$ and $(\check{\pi}_{s'},\mathscr{L}_{s'})$ for $s, s'\in\Theta(\varepsilon,\zeta)$, that is unique by Schur's lemma. Define

$$C(\varepsilon) := \{\sigma=(\sigma_1,\ldots,\sigma_r)\in\mathbb{C}^r;\ \sigma_k=0 \text{ for all } k \text{ such that } \varepsilon_k=0\}.$$

For $\sigma\in C(\varepsilon)$, we see from [12, Theorem 2.3(ii)] that the restriction χ_σ to the stabilizer $G(0)_{E^*_\varepsilon}$ at E^*_ε equals 1. Thus we can define a function $\Upsilon^\varepsilon_\sigma$ on $\mathcal{O}^*_\varepsilon$ by

$$\Upsilon^\varepsilon_\sigma(t\cdot E^*_\varepsilon) := \chi_\sigma(t) \qquad (t\in G(0)). \tag{4.9}$$

Then

$$\Upsilon^\varepsilon_\sigma(t\cdot\xi) = \chi_\sigma(t)\,\Upsilon^\varepsilon_\sigma(\xi) \qquad (t\in G(0),\ \xi\in\mathcal{O}^*_\varepsilon). \tag{4.10}$$

Conversely, any $G(0)$-relatively invariant function on $\mathcal{O}^*_\varepsilon$ is expressed as a constant multiple of $\Upsilon^\varepsilon_\sigma$ for some $\sigma\in C(\varepsilon)$. Noting $(\bar{s}'-\bar{s})/2\in C(\varepsilon)$, we consider the function $\Upsilon^\varepsilon_{(\bar{s}'-\bar{s})/2}$ on $\mathcal{O}^*_\varepsilon$. Since the measure $|\Upsilon^\varepsilon_{(\bar{s}'-\bar{s})/2}|^2\,dv_{s'} = \Upsilon^\varepsilon_{\mathfrak{R}s'-\mathfrak{R}s}\,dv_{s'}$ has the same relative invariance as dv_s by (4.6) and (4.10), there exists a positive constant $C_{s,s'}$ such that

$$dv_s = C_{s,s'}\,|\Upsilon^\varepsilon_{(\bar{s}'-\bar{s})/2}|^2\,dv_{s'}. \tag{4.11}$$

Then we have an unitary isomorphism $\Psi_{s,s'}\colon\mathscr{L}_s\to\mathscr{L}_{s'}$, where

$$\Psi_{s,s'}f(\xi,\cdot) := (C_{s,s'})^{1/2}\,\Upsilon^\varepsilon_{(\bar{s}'-\bar{s})/2}(\xi)\,f(\xi,\cdot) \qquad (f\in\mathscr{L}_s,\ \xi\in\mathcal{O}^*_\varepsilon). \tag{4.12}$$

PROPOSITION 4.5. *For elements s and s' of $\Theta(\varepsilon,\zeta)$, the map $\Phi_{s,s'}$ is the unitary intertwining operator between $(\check{\pi}_s,\mathscr{L}_s)$ and $(\check{\pi}_{s'},\mathscr{L}_{s'})$.*

Proof. It is sufficient to show that $\check{\pi}_{s'}(g) \circ \Psi_{s,s'} f = \Psi_{s,s'} \circ \check{\pi}_s(g) f$ ($g \in G$, $f \in \mathcal{L}_s$). For $g = n \in N(Q)$ and $\xi \in \mathcal{O}_\varepsilon^*$, we have by (4.8)

$$\check{\pi}_{s'}(n) \circ \Psi_{s,s'} f(\xi, \cdot) = (C_{s,s'})^{1/2}\, \Upsilon^{\varepsilon}_{(\bar{s}'-\bar{s})/2}(\xi)\, \tau_\xi(n)\, f(\xi, \cdot) = \Psi_{s,s'} \circ \check{\pi}_s(n) f(\xi, \cdot).$$

On the other hand, for $g = t \in G(0)$, $\xi \in \mathcal{O}_\xi^*$ and $u \in \mathfrak{g}(1/2)$, we see from (4.7) and (4.10) that

$$\begin{aligned}\check{\pi}_{s'}(t) \circ \Psi_{s,s'} f(\xi, u) &= \chi_{\bar{s}'/2}(t) \cdot (C_{s,s'})^{1/2}\, \Upsilon^{\varepsilon}_{(\bar{s}'-\bar{s})/2}(t^{-1} \cdot \xi)\, f(t^{-1} \cdot \xi, t^{-1} \cdot u)\\ &= (C_{s,s'})^{1/2}\, \Upsilon^{\varepsilon}_{(\bar{s}'-\bar{s})/2}(\xi) \cdot \chi_{\bar{s}/2}(t)\, f(t^{-1} \cdot \xi, t^{-1} \cdot u)\\ &= \Psi_{s,s'} \circ \check{\pi}_s(t) f(\xi, u).\end{aligned}$$

Therefore the statement is verified. ∎

4.3. *Determinant Type Polynomials Associated to the Dual Cone and the Corresponding Differential Operators*

Let us describe the clan $(\mathfrak{g}(1)^*, \triangle')$ of the dual cone $\Omega^* \subset \mathfrak{g}(1)^*$ with $E^* = E^*_{(1, \ldots, 1)}$ taken as the base point. First of all, we note that the bilinear form B on $\mathfrak{g}(1)$ given by $B(x, x') = \langle [jx', x], E^* \rangle$ $(x, x' \in \mathfrak{g}(1))$ is positive definite (see [12, Remark 1.3]). Thus, for $\xi \in \mathfrak{g}(1)^*$, there exists a unique $\check{\xi} \in \mathfrak{g}(1)$ such that $B(x, \check{\xi}) = \langle x, \xi \rangle$ $(x \in \mathfrak{g}(1))$, which means $\xi = E^* \circ \mathrm{ad}(j\check{\xi})$. Therefore the product $\triangle'$ is given by $\xi \triangle' \eta := \eta \circ \mathrm{ad}(j\check{\xi}) \in \mathfrak{g}(1)^*$ $(\xi, \eta \in \mathfrak{g}(1)^*)$. Let $\mathfrak{E}_k$ be the element of $\mathfrak{g}(1)^*$ defined by $\langle x, \mathfrak{E}_k \rangle := x_{r+1-k, r+1-k}$ for $x = \sum_{k=1}^r x_{kk} E_k + \sum_{m>k} X_{mk}$. Then it is deduced from [12, Sect. 2] that $\mathfrak{E}_k$ $(k = 1, \ldots, r)$ are idempotents, and that the clan $(\mathfrak{g}(1)^*, \triangle')$ allows a normal decomposition with respect to these $\mathfrak{E}_k$'s (cf. [20, Chap. 3, Sect. 6]). Let $\mathfrak{E}_I$ be the idempotent $\sum_{i \in I} \mathfrak{E}_i \in \mathfrak{g}(1)^*$. Denote by Ω_I^* the subcone corresponding to $\mathfrak{E}_I$, and let Δ_I^* and D_I^* be the reduced and composite determinant associated to Ω_I^* respectively. We extend Δ_I^* to the polynomial function on $\mathfrak{g}(1)^*$ as in Section 2. Applying Theorem 3.2, we see that

$$G(0) \cdot \mathfrak{E}_I = \left\{ \xi \in \mathfrak{g}(1)^*;\ \begin{array}{ll} \Delta^*_{I_\alpha}(\xi) > 0 & (\alpha = 1, \ldots, d) \\ \Delta^*_J(\xi) = 0 & (J \in \mathcal{M}'(I)) \end{array} \right\}, \tag{4.13}$$

where $d := \sharp I$ and $M'(I)$ is the set

$$\{I^{kk};\ k \notin I\} \cup \{I^{km};\ k \notin I,\ m > k,\ D^*_{I^{km}} \text{ is not divisible by } \Delta^*_{I^{kk}}\},$$

see the remark preceding Theorem 3.2.

Let $I(\varepsilon)$ be the subset $\{i;\ \varepsilon_i = 1\}$ of $\lfloor 1, r \rfloor$. When $I(\varepsilon) = \{i_1, i_2, \ldots, i_d\}$ with $1 \leqslant i_1 < \cdots < i_d \leqslant r$, we put $I(\varepsilon; \alpha) := \{i_\alpha, i_{\alpha+1}, \ldots, i_d\}$. For a subset I of

$\lfloor 1, r \rfloor$, we denote by I^* the set $\{r+1-i;\, i \in I\}$. Then we have $E_\varepsilon^* = \mathfrak{E}_{I(\varepsilon)^*}$ and (4.13) is rewritten as

$$\mathcal{O}_\varepsilon^* = \left\{ \xi \in \mathfrak{g}(1)^* ;\ \begin{array}{ll} \Delta^*_{I(\varepsilon;\,\alpha)^*}(\xi) > 0 & (\alpha = 1, ..., d) \\ \Delta^*_{J^*}(\xi) = 0 & (J \in \mathcal{M}^*(\varepsilon)) \end{array} \right\}, \tag{4.14}$$

where $\mathcal{M}^*(\varepsilon) := \{J^*;\, J \in \mathcal{M}'(I(\varepsilon)^*)\}$. Thanks to Theorem 2.3, $\Delta^*_{I(\varepsilon;\,\alpha)^*}$ is $G(0)$-relatively invariant on the orbit $\mathcal{O}_\varepsilon^*$, so that there exists $\mu(\varepsilon;\alpha) \in C(\varepsilon)$ for which

$$\Delta^*_{I(\varepsilon;\,\alpha)^*}(\xi) = \Upsilon^\varepsilon_{-\mu(\varepsilon;\,\alpha)}(\xi) \qquad (\xi \in \mathcal{O}_\varepsilon^*). \tag{4.15}$$

For a polynomial ϕ on $\mathfrak{g}(1)^*$, let $\phi(\partial_z)$ be the differential operator on $\mathfrak{g}(1)_{\mathbb{C}}$ such that $\phi(\partial_z)\, e^{\langle z, \xi \rangle} = \phi(\xi)\, e^{\langle z, \xi\rangle}$ $(\xi \in V^*)$. Recalling Proposition 4.4, we have for $f \in \mathcal{L}_s$ $(s \in \Theta(\varepsilon, \zeta))$ and $F := \Phi_s f \in \mathcal{H}_s(\mathcal{D})$,

$$\phi(-i\partial_z)\, F(z, u) = \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z, \xi\rangle} \phi(\xi)\, f(\xi, u)\, dv_s(\xi) \qquad ((z, u) \in D). \tag{4.16}$$

Based on these observations, we arrive at the following theorem, which can be regarded as a solvable version of results in [14, 15].

THEOREM 4.6. (i) *Each element F of $\mathcal{H}_s(\mathcal{D})$ satisfies the differential equations*

$$\Delta^*_{J^*}(-i\partial_z)\, F(z, u) = 0 \qquad (J \in \mathcal{M}^*(I)).$$

(ii) *The differential operator $\Delta^*_{I(\varepsilon;\,\alpha)^*}(-i\partial_z)$ $(\alpha = 1, ..., d)$ induces an intertwining map from $(\pi_s, \mathcal{H}_s(\mathcal{D}))$ to $(\pi_{s+2\mu(\varepsilon;\,\alpha)}, \mathcal{H}_{s+2\mu(\varepsilon;\,\alpha)}(\mathcal{D}))$.*

(iii) *Let s, s' be elements of $\Theta(\varepsilon, \zeta)$. If an intertwining map from $(\pi_s, \mathcal{H}_s(\mathcal{D}))$ to $(\pi_{s'}, \mathcal{H}_{s'}(\mathcal{D}))$ is induced by a differential operator Z, the same intertwining map is also given by a product of powers of $\Delta^*_{I(\varepsilon;\,\alpha)^*}(-i\partial_z)$ $(\alpha = 1, ..., d)$ up to constant multiple.*

We remark that the assertion (iii) does not state that the differential operator Z itself is expressed as a product of powers of $\Delta^*_{I(\varepsilon;\,\alpha)^*}(-i\partial_z)$.

Proof. (i) Clear from (4.16) and (4.14).

(ii) First of all, we show that $s + 2\mu(\varepsilon;\alpha) \in \Theta(\varepsilon;\zeta)$. Since $\mu(\varepsilon;\alpha)$ belongs to $C(\varepsilon)$, it suffices to show that each component of $\mu(\varepsilon;\alpha)$ is non-negative (see (4.4)). Express $\mu(\varepsilon;\alpha)$ as $(\mu_1, \mu_2, ..., \mu_r)$. Let τ be an element $\exp(\sum_{k=1}^r (\log \tau_k)\, A_k)$ of $G(0)$ with $\tau_1, ..., \tau_r \in \mathbb{R}$. Then we see from (4.15), (4.9) and (4.2) that $\Delta^*_{I(\varepsilon;\,\alpha)^*}(\tau \cdot E_\varepsilon^*) = \chi_{-\mu(\varepsilon;\,\alpha)}(\tau) = \prod_{k=1}^r \tau_k^{-\mu_k}$. On the other hand, putting $E_k^* = \mathfrak{E}_{r+1-k}$, we have $\tau \cdot E_\varepsilon^* = \sum_{k=1}^r \varepsilon_k \tau_k^{-1} E_k^*$ by virtue of [12, Sect. 2]. Thus we get

$$\Delta^*_{I(\varepsilon;\,\alpha)^*}\left(\sum_{k=1}^r \varepsilon_k \tau_k^{-1} E_k^*\right) = \prod_{k=1}^r \tau_k^{-\mu_k},$$

which implies that the μ_k's are non-negative integers because $\Delta^*_{I(\varepsilon;\,\alpha)^*}$ is a polynomial function. Therefore we conclude $s + 2\mu(\varepsilon;\alpha) \in \Theta(\varepsilon;\zeta)$. Putting $s' := s + 2\mu(\varepsilon;\alpha)$, we have $\mu(\varepsilon;\alpha) = (s'-s)/2$. Then we see from (4.16), (4.15), (4.11), and (4.12) that

$$\begin{aligned}
&\Delta^*_{I(\varepsilon;\,\alpha)^*}(-i\partial_z) \circ \Phi_s f(z,u)\\
&\quad = \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z,\xi\rangle} \Upsilon^\varepsilon_{(s-s')/2}(\xi) f(\xi,u)\, dv_s(\xi)\\
&\quad = C_{s,s'} \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z,\xi\rangle} \Upsilon^\varepsilon_{(\bar{s}'-\bar{s})/2}(\xi) f(\xi,u)\, dv_{s'}(\xi)\\
&\quad = (C_{s,s'})^{1/2}\, \Phi_{s'} \circ \Psi_{s,s'} f(z,u) \qquad (f \in \mathcal{L}_s,\ (z,u) \in \mathcal{D}). \qquad (4.17)
\end{aligned}$$

Therefore the assertion holds thanks to Proposition 4.5.

(iii) Proposition 4.5 together with Schur's lemma tells us that the intertwining map must be transferred to $\Psi_{s,s'} : \mathcal{L}_s \to \mathcal{L}_{s'}$ up to constant multiple by means of Φ_s and $\Phi_{s'}$. Namely we have for $f \in \mathcal{L}_s$

$$Z \circ \Phi_s f(z,u) = C \cdot \Phi_{s'} \circ \Psi_{s,s'} f(z,u)$$

with some constant C. The left-hand side is rewritten as

$$Z\left[\int_{\mathcal{O}_\varepsilon^*} e^{i\langle z,\xi\rangle} f(\xi,u)\, dv_s(\xi)\right],$$

and the same calculation as in (4.17) shows that the right-hand side equals

$$C(C_{s,s'})^{-1/2} \int_{\mathcal{O}_\varepsilon^*} e^{i\langle z,\xi\rangle} \Upsilon^\varepsilon_{(s-s')/2}(\xi) f(\xi,u)\, dv_s(\xi).$$

Comparison of these expressions implies that $\Upsilon^\varepsilon_{(s-s')/2}$ can be extended to a polynomial function on V^*. Hence the assertion follows from Theorem 2.3. ∎

Remark 4.7. For a subset I of $\lfloor 1, r \rfloor$, the cone Ω^*_{I*} is dual to the subcone $\Omega^I \subset \mathfrak{g}(1)^I := \sum^{\oplus}_{k \in I} \mathbb{R}E_k \oplus \sum^{\oplus}_{m,\, k \in I} \mathfrak{g}_{(\alpha_m + \alpha_k)/2}$. When we identify the polynomial algebra $\mathscr{P}(\mathfrak{g}(1)^*)$ on $\mathfrak{g}(1)^*$ with the symmetric algebra $\mathscr{S}(\mathfrak{g}(1))$ of $\mathfrak{g}(1)$, the element Δ^*_{I*} belongs to the algebra $\mathscr{S}(\mathfrak{g}(1)^I)$. If Ω is symmetric, Ω^*_{I*} and Δ^*_{I*} are naturally identified with Ω^I and Δ^I, respectively.

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